hep-ph/9604270
April 1996

The sunset diagram in SU(3) chiral perturbation theory

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Abstract

A general procedure for the calculation of a class of two-loop Feynman diagrams is described. These are two-point functions containing three massive propagators, raised to integer powers, in the denominator, and arbitrary polynomials of the loop momenta in the numerator. The ultraviolet divergent parts are calculated analytically, while the remaining finite parts are obtained by a one-dimensional numerical integration, both below and above the threshold. Integrals of this type occur, for example, in chiral perturbation theory at order p^6.

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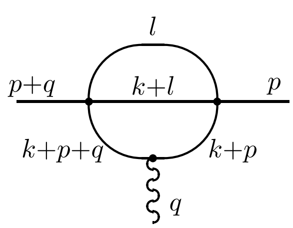
1 Introduction

During the last couple of years there have been growing efforts to extend the predictions of chiral perturbation theory, which is an effective theory of QCD at low energies, to the order $\mathcal{O}(p^6)$ of the momentum expansion. Of special interest are processes which vanish at $\mathcal{O}(p^2)$: examples such as $\gamma\gamma \to \pi^0\pi^0$ [1] or $\eta \to \pi^0\gamma\gamma$ [2] show that the $\mathcal{O}(p^6)$ contributions can be rather large but necessary to find agreement between the theoretical predictions and experimental data. In going beyond $\mathcal{O}(p^4)$ one is confronted with two major difficulties. On the one hand, the chirally invariant structures of order $\mathcal{O}(p^6)$ [3] come with a number of a priori unknown constants which are to be determined from experiment or estimated in some phenomenological model. On the other hand, at order $\mathcal{O}(p^6)$, the diagrammatic expansion involves two-loop Feynman integrals. In our paper we focus on the second point and discuss a class of two-loop Feynman diagrams which arise naturally in the context of SU(3) chiral perturbation theory, namely integrals of the *sunset* topology for *arbitrary internal masses, tensor numerators* and (for reasons which will become clear in a moment) *powers of denominators.* The necessity to consider the arbitrary mass case is a characteristic distinction between SU(2) and SU(3) chiral perturbation theory. It is due to the presence (in the limit of isospin conservation) of three different masses (m_π, m_K, m_η) in the latter, as opposed to only one in the former (m_π). As a consequence, the calculation of the genuine two-loop diagrams becomes essentially more complex.

Any field theory containing four-point interaction vertices with derivative couplings produces sunset-type integrals with numerators at the two-loop level. This is obvious for self-energies, but is also the case, for example, if one studies form factors for small momentum transfer, such as electromagnetic charge-radii of mesons. A typical contribution is represented by the following diagram (the wiggly line is a photon, the others are mesons):



(1)

Taylor expanding the corresponding Feynman integral with respect to the momentum transfer q around $q=0$ generates additional loop momenta in the numerator, increases the power of one denominator, and reduces the original three-point function to a two-point function. The resulting integrals in $D=4-2\epsilon$ dimensions [4] are of the form

$$\int d^D k \, d^D l \, \frac{(k\cdot p)^i (k\cdot q)^j (l\cdot p)^k (l\cdot q)^l}{P^n_{k+p,m_1} P_{k+l,m_2} P_{l,m_3}} \tag{2}$$

with integer i, j, k, l, n and propagators $P_{k,m} = k^2 - m^2 + i\eta$. After a tensor decomposition with respect to q and expressing the remaining scalar products of k, l and p in terms of P_{k+p,m_1}, P_{k+l,m_2}, P_{l,m_3} and the Mandelstam-type variables

$$s_{12} = (p-l)^2 \tag{3}$$
$$s_{23} = k^2, \tag{4}$$

(2) can be reduced to a combination of the following basic sunset integrals

$$T_{\alpha,\beta,n_1,n_2,n_3}\left(m_1^2,m_2^2,m_3^2;p^2\right) \quad = \quad \pi^{-D}\int d^Dk\, d^Dl\, \frac{s_{12}^{\alpha}\, s_{23}^{\beta}}{P_{k+p,m_1}^{n_1} P_{k+l,m_2}^{n_2} P_{l,m_3}^{n_3}} \tag{5}$$

and more simple graphs where one of the propagators has been cancelled. The latter are essentially products of well known one-loop vacuum integrals [5] and thus easy to calculate, whereas the former are not known analytically except for special combinations of masses. The calculation of the quadratic and quartic charge-radii of the light pseudoscalar mesons in SU(3) chiral perturbation theory to the order $\mathcal{O}(p^6)$, for instance, requires the general nonzero mass case of $T_{\alpha,\beta,n_1,1,1}$ with $\alpha+\beta \leq 7$ and $n_1 \leq 6$ and illustrates that rather high tensors and powers of denominators actually occur in practice [6].

Sunset diagrams without numerators ($\alpha=\beta=0$) have been studied in several papers. In [7] they were reduced, by direct integration in momentum space [8], to two-dimensional integrals, which were then evaluated numerically. A one-dimensional integral representation, derived from a Feynman parametrization, was used in [9]. Another one-dimensional integral representation was obtained from a dispersion relation in [10, 11]. For the case of equal masses, series expansions in p^2 were considered in [12, 13], and for the general mass case, various multiple series in p^2 and the masses m_i^2 were given in [11, 14]. In [15], it was shown that the sunset diagram is a solution of a sixth order ordinary differential equation in p^2.

Less is known about the above integrals $T_{\alpha,\beta,n_1,n_2,n_3}$ for positive α, β. It is possible to regard them as master integrals (i.e. two-point functions with five propagators) where two of the propagators are massless and are raised to negative powers. In this way, expansions in p^2 and $1/p^2$ can be calculated by the recursive procedures described in [16]. However, as one goes to higher orders in the expansions, the complexity of the coefficients increases very rapidly. Therefore, in practice, the number of terms that can be obtained is limited, and may or may not be sufficient, depending on the accuracy required and the value of p^2 considered.

In this paper we present a hybrid method for calculating the integrals $T_{\alpha,\beta,n_1,n_2,n_3}$ with arbitrary nonzero masses. First, we split off a Taylor polynomial in p^2, which enables us to calculate the ultraviolet divergent part in an exact analytical form. Then, using a dispersion relation, the remaining part, which is finite as $D\to 4$, is expressed as a one-dimensional integral which can be evaluated numerically to any desired precision. It is a generalization of the integral representation for $T_{0,0,1,1,1}$ given in [10, 11].

Our paper is divided into three parts: in section 2 the method is described in some detail, in section 3 the structure of the one-dimensional integration is analyzed with respect to numerical instabilities, and in section 4 relations among the $T_{\alpha,\beta,n_1,n_2,n_3}$ are established, which can be used to minimize the number of evaluations of these integrals required in the calculation of a physical process.

2 Presentation of the method

An important property of diagrams of the sunset topology is that their ultraviolet divergences (both the overall divergence, and those due to divergent subdiagrams) are polynomials in the external momentum p^2. Thus, they can easily be isolated by performing

a Taylor expansion in this variable. Because we consider only diagrams with massive propagators, infrared divergences cannot occur.

In the following discussion, the integrals (5) will be written as $T(p^2)$ for simplicity. They can be continued analytically into the complex p^2-plane, with a branch cut along the positive real axis from $p^2=(m_1+m_2+m_3)^2$ to infinity. For any p^2 not on this cut, we may apply Cauchy's theorem, keeping $D\neq 4$ to regulate possible ultraviolet divergences,

$$(1-\mathcal{T}^{(r)})T(p^2) = \frac{1}{2\pi i}\int_C d\zeta\, \frac{(p^2)^r}{(\zeta-p^2)\zeta^r}T(\zeta) \tag{6}$$

where the operator $\mathcal{T}$ extracts the first r terms of the Taylor expansion in p^2,

$$\mathcal{T}^{(r)} = \sum_{j=0}^{r-1}\frac{(p^2)^j}{j!}\frac{\partial^j}{(\partial p^2)^j}\bigg|_{p^2=0}, \tag{7}$$

and C is any contour in the complex plane that encloses the points $\zeta=0$ and $\zeta=p^2$ and avoids the cut. On the cut, $T(p^2)$ can be recovered from (6) by letting p^2 approach the real p^2-axis from above.

For C we choose a path consisting of two straight lines, one along each side of the cut, connected by an infinitesimally small circle around the branch point $\zeta=(m_1+m_2+m_3)^2$, and a large circle with radius R. If the number of subtractions r is sufficiently large (this will be specified more precisely later), the contribution from the large circle will vanish in the limit $R\to\infty$, yielding

$$(1-\mathcal{T}^{(r)})T(p^2) = \frac{1}{2\pi i}\int_{(m_1+m_2+m_3)^2}^{\infty} d\zeta\, \frac{(p^2)^r}{(\zeta-p^2)\zeta^r}\{T(\zeta+i0)-T(\zeta-i0)\}\,. \tag{8}$$

At this stage, the left and right hand sides of eq. (8) are still dimensionally regularized. However, the quantity $\{T(\zeta+i0)-T(\zeta-i0)\}=2iImT(\zeta)$ is finite in 4 dimensions, and therefore, if r is chosen large enough, the integral on the right hand side may be evaluated with $D=4$. Thus we decompose $T(p^2)$ into

$$T(p^2) = T^A(p^2)+T^N(p^2)\,, \tag{9}$$

where

$$T^A(p^2) = \mathcal{T}^{(r)}T(p^2) \tag{10}$$

contains all the ultraviolet divergences and will be calculated analytically, and

$$T^N(p^2) = \frac{1}{\pi}\int_{(m_1+m_2+m_3)^2}^{\infty} d\zeta\, \frac{(p^2)^r}{(\zeta-p^2)\zeta^r}ImT(\zeta) \tag{11}$$

is finite in $D=4$ and will be calculated numerically.

In order to calculate the analytic part (10), we express the derivatives of $T(p^2)$ at $p^2=0$ in terms of derivatives with respect to a parameter ρ, by which we multiply the momentum p:

$$\frac{(p^2)^j}{j!}\frac{\partial^j T(p^2)}{(\partial p^2)^j}\bigg|_{p^2=0} = \frac{1}{(2j)!}\frac{\partial^{2j}T((\rho p)^2)}{\partial\rho^{2j}}\bigg|_{\rho=0}. \tag{12}$$

These can be evaluated by differentiation underneath the integrals in (5). When ρ is subsequently set equal to zero, the external momentum p disappears from the propagators, leaving a sum of vacuum integrals of the form

$$\pi^{-D} \int d^D k\, d^D l \, \frac{(k\cdot p)^{\alpha'} (l\cdot p)^{\beta'}}{(k^2-m_1^2)^{n'_1}((k+l)^2-m_2^2)^{n'_2}(l^2-m_3^2)^{n'_3}}. \tag{13}$$

Such integrals can be calculated recursively for arbitrary integers α', β', n'_i [17, 18, 16].

Let us now consider the dispersion integral (11) in the special case $n_1=n_2=n_3=1$. More general cases will be obtained later by differentiation with respect to the masses. To determine the minimal number of subtractions r required, we need some information about the behaviour of $T(p^2)$ as $|p^2|\to\infty$, which can be obtained from general theorems on asymptotic expansions of Feynman diagrams [19]. For our purposes, it is enough to know that, in $D=4-2\epsilon$ dimensions, the large p^2 expansion of $T_{\alpha,\beta,1,1,1}$ has the following structure:

$$T_{\alpha,\beta,1,1,1}(p^2) = (-p^2)^{\alpha+\beta+1-2\epsilon} \sum_{\sigma=0}^{\infty} \frac{a_\sigma}{(p^2)^\sigma} + (-p^2)^{\alpha+\beta-\epsilon} \sum_{\sigma=0}^{\infty} \frac{b_\sigma}{(p^2)^\sigma} + (-p^2)^{\alpha+\beta-1} \sum_{\sigma=0}^{\infty} \frac{c_\sigma}{(p^2)^\sigma} \,. \tag{14}$$

This shows that, for ϵ close to zero, the contribution from the large circle to (6) will vanish in the limit $R\to\infty$ if we choose $r\geq\alpha+\beta+2$. Although the calculations are simplest when r is chosen minimally ($r=\alpha+\beta+2$), we are free to use a larger value, which amounts to shifting an additional finite contribution from T^N to T^A. This provides us with an important consistency check on the method.

An application of Cutkosky's rules [20] gives the imaginary part of $T_{\alpha,\beta,1,1,1}$ as a two dimensional integral over three body phase space (the Dalitz plot):

$$Im T_{\alpha,\beta,1,1,1}(p^2) \;=\; \frac{\pi}{p^2} \int\limits_{\Omega(p^2)} ds_{12} ds_{23} s_{12}^{\alpha} s_{23}^{\beta} \,, \tag{15}$$

where the integration region $\Omega(p^2)$ is given by

$$\begin{aligned} &4m_1^2m_2^2m_3^2 + (s_{12}-m_1^2-m_2^2)(s_{23}-m_2^2-m_3^2)(s_{13}-m_1^2-m_3^2) \\ &-m_3^2(s_{12}-m_1^2-m_2^2)^2 - m_1^2(s_{23}-m_2^2-m_3^2)^2 - m_2^2(s_{13}-m_1^2-m_3^2)^2 \;>\; 0 \end{aligned} \tag{16}$$

with

$$s_{12} > (m_1+m_2)^2 \,, \quad s_{23} > (m_2+m_3)^2 \,, \quad s_{13} > (m_1+m_3)^2 \,. \tag{17}$$

In (16)-(17), $s_{13}=(k+p+l)^2$ depends on s_{12} and s_{23} as follows:

$$s_{13} = p^2 + m_1^2 + m_2^2 + m_3^2 - s_{12} - s_{23} \,. \tag{18}$$

One of the integrations in (15) can be performed easily; e.g., at fixed s_{23}, the range of s_{12} is $A-B<s_{12}<A+B$ with

$$A \;=\; \frac{1}{2s_{23}}\Big(p^2(m_2^2-m_3^2+s_{23}) - s_{23}^2 + s_{23}(m_1^2+m_2^2+m_3^2) - m_1^2(m_2^2-m_3^2)\Big) \tag{19}$$

$$B \;=\; \frac{1}{2s_{23}}\sqrt{\lambda(s_{23},p^2,m_1^2)\,\lambda(s_{23},m_2^2,m_3^2)}\,, \tag{20}$$

where $\lambda(x,y,z)=(x-y-z)^2-4yz$ is the Källén function, so that

$$ImT_{\alpha,\beta,1,1,1}(p^2)=\frac{\pi}{p^2}\,\theta(p^2-(m_1+m_2+m_3)^2)\int\limits_{(m_2+m_3)^2}^{(p-m_1)^2}ds_{23}\,\frac{2s_{23}^{\beta}}{\alpha+1}\sum_{j=0}^{\left[\frac{\alpha}{2}\right]}\binom{\alpha+1}{2j+1}A^{\alpha-2j}B^{2j+1}\,. \tag{21}$$

In general, performing the second integration gives complete elliptic integrals [10]. However, it is not necessary to evaluate them explicitly here. Instead, we can insert the result (21) into the dispersion integral (11) and interchange the $d\zeta$ and ds_{23} integrations:

$$\begin{aligned}(1-\mathcal{T}^{(r)})\,T_{\alpha,\beta,1,1,1}(p^2) \;=\; & \int\limits_{(m_2+m_3)^2}^{\infty}ds_{23}\;s_{23}^{\beta-1}\sqrt{\lambda(s_{23},m_2^2,m_3^2)}\\ & \cdot\int\limits_{(\sqrt{s_{23}}+m_1)^2}^{\infty}d\zeta\;\frac{(p^2)^r}{(\zeta-p^2)\,\zeta^r}\,\frac{\sqrt{\lambda(\zeta,m_1^2,s_{23})}}{\zeta}P(\zeta)\,,\end{aligned} \tag{22}$$

where

$$P(\zeta)\;=\;\frac{1}{\alpha+1}\sum_{j=0}^{\left[\frac{\alpha}{2}\right]}\binom{\alpha+1}{2j+1}A^{\alpha-2j}B^{2j}\Big|_{p^2=\zeta} \tag{23}$$

is a polynomial in ζ of degree α. The ζ-integrals in (22) can be recognized as subtracted dispersion integral representations of the one-loop scalar two-point function,

$$B(m_1^2,m_2^2;p^2)\;=\;-i\pi^{-D/2}\int d^Dk\;\frac{1}{((k+p)^2-m_1^2)(k^2-m_2^2)}\,, \tag{24}$$

in which one of the masses is replaced with s_{23}:

$$(1-\mathcal{T}^{(k)})\,B(m_1^2,s_{23};p^2)\;=\;\int\limits_{(\sqrt{s_{23}}+m_1)^2}^{\infty}d\zeta\;\frac{(p^2)^k}{(\zeta-p^2)\,\zeta^k}\,\frac{\sqrt{\lambda(\zeta,m_1^2,s_{23})}}{\zeta}\,. \tag{25}$$

Thus, the numerical part $T^N_{\alpha,\beta,1,1,1}(p^2)$ is given by a sum of one-dimensional integrals of the form

$$\int\limits_{(m_2+m_3)^2}^{\infty}ds_{23}\;s_{23}^{l}\sqrt{\lambda(s_{23},m_2^2,m_3^2)}\;\left(1-\mathcal{T}^{(k)}\right)B(m_1^2,s_{23};p^2)\,, \tag{26}$$

with

$$k\;=\;r-\alpha,\,r-\alpha+1,\;\ldots\;,r \tag{27}$$
$$l\;=\;\beta-\alpha-1,\,\beta-\alpha,\;\ldots\;,\beta+\alpha-1+k-r. \tag{28}$$

We use the following well known expressions for $B(m_1^2,s_{23};p^2)$ [5]:

$$\begin{aligned}B(m_1^2,s_{23};p^2)\;&=\;\frac{1}{\epsilon}-\gamma-\int_0^1 dx\log(m_1^2x+s_{23}(1-x)-p^2x(1-x)) && (29)\\ &=\;\frac{1}{\epsilon}-\gamma+2-\log m_1^2+F(m_1^2;s_{23};p^2)\,, && (30)\end{aligned}$$

where

$$F(m_1^2; s_{23}; p^2) \;=\; x_1 \log\left(1 - \frac{1}{x_1}\right) + x_2 \log\left(1 - \frac{1}{x_2}\right) , \tag{31}$$

$x_{1,2}$ are the zeros of the argument of the logarithm in the Feynman parameter integral,

$$x_{1,2} \;=\; \frac{1}{2p^2}\left\{p^2 + s_{23} - m_1^2 \pm \sqrt{\lambda(p^2, s_{23}, m_1^2)}\right\} , \tag{32}$$

and γ is Euler's constant. The derivatives at $p^2 = 0$ that enter (26) can easily be obtained from (29), e.g.:

$$B(m_1^2, s_{23}; 0) \;=\; \frac{1}{\epsilon} - \gamma + 1 - \log m_1^2 + \frac{s_{23}}{m_1^2 - s_{23}} \log \frac{s_{23}}{m_1^2} \tag{33}$$

$$\frac{\partial B}{\partial p^2}(m_1^2, s_{23}; 0) \;=\; \frac{m_1^2 + s_{23}}{2(m_1^2 - s_{23})^2} + \frac{m_1^2 s_{23}}{(m_1^2 - s_{23})^3} \log \frac{s_{23}}{m_1^2} . \tag{34}$$

Integrals $T_{\alpha,\beta,n_1,n_2,n_3}$ where one or more of the indices n_1, n_2, n_3 are greater than one are related to $T_{\alpha,\beta,1,1,1}$ by differentiation with respect to m_1^2, m_2^2, m_3^2:

$$\begin{aligned} & T_{\alpha,\beta,n_1+1,n_2+1,n_3+1}\left(m_1^2, m_2^2, m_3^2; p^2\right) \\ &= \frac{1}{n_1! n_2! n_3!} \frac{\partial^{n_1+n_2+n_3}}{(\partial m_1^2)^{n_1} (\partial m_2^2)^{n_2} (\partial m_3^2)^{n_3}} T_{\alpha,\beta,1,1,1}\left(m_1^2, m_2^2, m_3^2; p^2\right) . \end{aligned} \tag{35}$$

Differentiating $T^A_{\alpha,\beta,1,1,1}$ in this manner leads to vacuum integrals of the same kind as in (13). For the numerical part, $T^N_{\alpha,\beta,n_1,n_2,n_3}$, it is necessary to differentiate the integrals (26). This can be done underneath the integral sign, provided one first shifts the integration variable $s_{23} \to s'_{23} + (m_2 + m_3)^2$, in order to avoid difficulties due to the square root like behaviour of the integrand at the lower end point. Furthermore, the mass derivatives of the one-loop function B are calculated most effectively using the fact that the nontrivial part F essentially reproduces itself:

$$\frac{\partial B(m_1^2, s_{23}; p^2)}{\partial m_1^2} = \frac{1}{\lambda(m_1^2, s_{23}, p^2)} \left[(m_1^2 - s_{23} - p^2) F(m_1^2; s_{23}; p^2) + 2 s_{23} \log \frac{m_1^2}{s_{23}}\right] . \tag{36}$$

3 Numerical analysis

When we are below the physical threshold, i.e. $p^2 < (m_1 + m_2 + m_3)^2$, the integrands in (26) are smooth and their numerical integration does not present any problems. However, some care is needed to evaluate the integrands with sufficient accuracy. The difficulties come from the following regions:

$\boldsymbol{s_{23} \to \infty}$**:** The expansion of $B(m_1^2, s_{23}; p^2)$ in $1/s_{23}$ reads (see also [14, 10]):

$$B(m_1^2, s; p^2) = \frac{1}{\epsilon} - \gamma + 1 - \log m_1^2 + \sum_{\sigma=1}^{\infty} \frac{1}{s_{23}^{\sigma}} \sum_{k=0}^{\sigma} (p^2)^k (m_1^2)^{\sigma-k} \left[a_{k\sigma} \log \frac{m_1^2}{s_{23}} + b_{k\sigma}\right] , \tag{37}$$

where

$$\begin{aligned}
a_{k\sigma} &= \frac{\sigma!(\sigma-1)!}{k!(k+1)!(\sigma-k)!(\sigma-k-1)!} \quad \text{for } k=0,\,\ldots,\sigma-1 \\
a_{\sigma\sigma} &= 0 \\
b_{0\sigma} &= 0 \\
b_{k\sigma} &= a_{k\sigma}\{\psi(\sigma)+\psi(\sigma+1)-\psi(\sigma-k)-\psi(\sigma-k+1)\} \\
& \quad \text{for } k=1,\,\ldots,\sigma-1 \text{ and } \psi(n) = -\gamma+\tfrac{1}{1}+\tfrac{1}{2}+\,\ldots\,+\tfrac{1}{n-1} \\
b_{\sigma\sigma} &= \frac{1}{\sigma(\sigma+1)}\,.
\end{aligned} \tag{38}$$

By omitting all summands containing powers of p^2 less than r, it is seen that the leading term of $(1-\mathcal{T}^{(r)})B(p^2)$ is of order $1/s_{23}^r$. Consequently, using the exact formulae gives rise to large cancellations, and it is much better to replace the integrand by its asymptotic expansion whenever $s_{23} \geq S\,(m_1+\sqrt{p^2})^2$, $S = const.$ For example, we obtained numerically stable results with the choice $S = 4$ when we truncated the series (37) at $\sigma = 50$.

$\boldsymbol{s_{23} \approx m_1^2}$**:** If $m_2+m_3<m_1$, the region $s_{23}\approx m_1^2$ lies within the integration interval and causes problems due to factors of $(m_1^2-s_{23})$ in the denominator (see e.g. (33) f). Nevertheless, the singularities of the different parts exactly cancel: the integrand is smooth at $s_{23}\approx m_1^2$ and can be approximated by a Taylor-polynomial or rather a Padé-approximant. For integrals $T_{\alpha,\beta,n_1,1,1}$ with $\alpha+\beta\leq 7$ and $n_1\leq 6$ a diagonal Padé-approximation of order $(20,20)$ in $s_{23}\in[0.2m_1^2, 5m_1^2]$ turned out to be a suitable choice.

$\boldsymbol{s_{23} \approx (m_1 \pm \sqrt{p^2})^2}$**:** Differentiation of $B(m_1^2, s_{23}; p^2)$ with respect to m_1^2 (or s_{23}) brings down a factor $\lambda(m_1^2, s_{23}, p^2)$ in the denominator, which vanishes for $s_{23}=(m_1\pm\sqrt{p^2})^2$. Since B and its derivatives are smooth at $s_{23}=(m_1+\sqrt{p^2})^2$ and (below the physical threshold) at $s_{23}=(m_1-\sqrt{p^2})^2$, a simple Taylor-approximation (up to order 10, e.g.) remedies the instability.

As a rule, cancellations are more severe when higher mass derivatives are taken or more terms of the Taylor series need to be subtracted.

Above the threshold, $p^2>(m_1+m_2+m_3)^2$, singularities appear in the integrand at $s_{23}=(\sqrt{p^2}-m_1)^2$, where the threshold of $B(m_1^2, s_{23}; p^2)$ is crossed. Although B itself is still finite at this point, its derivatives are not. Therefore, noting that $B(m_1^2, s_{23}; p^2)$ is an analytic function of s_{23} in the lower half of the complex plane, we deform the integration contour so as to steer clear of the singular point. The simplest possibility is to choose a straight line,

$$s_{23} \;=\; (m_2+m_3)^2 + e^{-\mathrm{i}\theta} s'\,, \quad 0 < s' < \infty\,, \tag{39}$$

at some angle $0<\theta<\pi/2$ with respect to the real s_{23}-axis. Along the deformed contour the integrand is smooth and can be integrated easily[1]. The fact that the result should not depend on θ can be used as a check. Notice, that such a rotation of the contour also

[1]This approach was also used in [9], where similar problems were encountered.

avoids the problems at $s_{23} \approx m_1^2$ and at $s_{23} \approx (m_1 \pm \sqrt{p^2})^2$ mentioned above, unless the left end of the integration contour happens to be a critical point itself.

We compared a number of special cases with results available in the literature. We computed the first 17 Taylor coefficients of $T_{0,0,1,1,1}(m_1^2, m_2^2, m_3^2; p^2)$ and verified that they satisfy the five-term recurrence relation derived in [15]. Another check involved the three point function (1), which can be calculated analytically in the special case $p^2 = (p+q)^2 = m_1^2 = m_2^2 = m_3^2$ [21]. We expanded the result around $q^2 = 0$ and found agreement with the first 3 terms as calculated by the algorithm described in this paper. We also checked that our algorithm reproduces the ϵ^{-2}, ϵ^{-1} and ϵ^0 terms of $T_{0,0,1,1,1}(m^2, m^2, m^2; p^2 = m^2)$, which can be found in [13].

Finally, as an example, we would like to give some concrete results for $T_{0,3,4,1,1}$. We take the masses $m_1 = 0.28$, $m_2 = 1$, $m_3 = 1.143328474$, corresponding to the masses of the π, K and η_8 mesons in chiral perturbation theory (in units of m_K), and two values of p^2, one below ($p^2 = 1$) and one above ($p^2 = 9$) the threshold. The parameter r denotes the number of Taylor terms subtracted from $T_{0,3,4,1,1}$ and defines $T^A_{0,3,4,1,1} = \mathcal{T}^{(r)} T_{0,3,4,1,1}$, cf. (10). The divergent part of $T^A_{0,3,4,1,1}$ is given by

$$\begin{aligned} T^{A,div}_{0,3,4,1,1} \;=\; & -\frac{1}{2\epsilon^2}\left(6p^2 + 4m_1^2 + m_2^2 + m_3^2\right) + \frac{1}{6\epsilon}\Big(18p^2 - 10m_1^2 - 9m_3^2 - 9m_2^2 \\ & -\frac{(p^2)^3}{m_1^4} + 12\frac{(p^2)^2}{m_1^2} + 36\gamma p^2 + 24\gamma m_1^2 + 6\gamma m_2^2 + 6\gamma m_3^2 \\ & +36p^2 \log(m_1^2) + 24m_1^2 \log(m_1^2) + 6m_2^2 \log(m_2^2) + 6m_3^2 \log(m_3^2)\Big)\,. \end{aligned} \tag{40}$$

The finite (order ϵ^0) part of $T^A_{0,3,4,1,1}$, and $T^N_{0,3,4,1,1}$ are given in the following table for different values of r:

r	$T^N_{0,3,4,1,1}(1)$	$T^{A,fin}_{0,3,4,1,1}(1)$	$T^N_{0,3,4,1,1}(9)$	$T^{A,fin}_{0,3,4,1,1}(9)$
5	-0.2879782058	-43.6425974985	$3267.7085 - 38384.2371\,i$	-61507.6741
6	-0.0318842307	-43.8986914736	$18389.8016 - 38384.2371\,i$	-76629.7672
7	-0.0040709644	-43.9265047399	$33170.9116 - 38384.2371\,i$	-91410.8772
8	-0.0005618259	-43.9300138784	$49955.0126 - 38384.2371\,i$	-108194.9782
	$T^{fin}_{0,3,4,1,1}(1) = -43.9305757043$		$T^{fin}_{0,3,4,1,1}(9) = -58239.9656 - 38384.2371\,i$	

The bottom row shows the total finite part $T^{fin}_{0,3,4,1,1} = T^N_{0,3,4,1,1} + T^{A,fin}_{0,3,4,1,1}$, which is independent of r, as it should be. Below the threshold, the Taylor series converges, so that in cases where only moderate accuracy is required (eg., 4 digits), it may be possible to neglect the numerical contribution $T^N_{0,3,4,1,1}$. In our application [6], however, we sometimes needed as many as 8 digits, due to large cancellations between different $T_{\alpha,\beta,n_1,n_2,n_3}$'s. In any case, $T^N_{0,3,4,1,1}$ is indispensable above the threshold, where the Taylor series diverges.

4 Relations among the basic sunset integrals

The method presented in the previous section enables us to calculate analytically the divergent part and (for given masses and external momentum) numerically the finite part of each $T_{\alpha,\beta,n_1,n_2,n_3}$. Nevertheless, the question arises whether the set of $T_{\alpha,\beta,n_1,n_2,n_3}$ can

be reduced further, both for practical and for theoretical reasons. In fact, there are three sources of relations,

- permutation symmetry, if some masses are equal (e.g. $T_{\alpha,\beta,1,1,1} = T_{\beta,\alpha,1,1,1}$ if $m_1 = m_3$),
- integration by parts identities [4, 17, 22],
- subloop tensor decomposition,

the third of which we would like to exploit in the sequel.

The most nontrivial part of a fixed $T_{\alpha,\beta,n_1,n_2,n_3}$ is the integral

$$\int d^Dk\ d^Dl\ \frac{(l\cdot p)^\alpha\ (k^2)^\beta}{P^{n_1}_{k+p,m_1} P^{n_2}_{k+l,m_2} P^{n_3}_{l,m_3}} \quad = \quad \int d^Dk\ \frac{(k^2)^\beta}{P^{n_1}_{k+p,m_1}}\ p_{\mu_1}\dots p_{\mu_\alpha}\ \int d^Dl\ \frac{l^{\mu_1}\dots l^{\mu_\alpha}}{P^{n_2}_{k+l,m_2} P^{n_3}_{l,m_3}}\ . \tag{41}$$

Consider for a moment just the l-integration and decompose its tensor structure in terms of $g^{\mu\nu}$ and k^μ, which is the external momentum of the l-subloop [23]. This yields

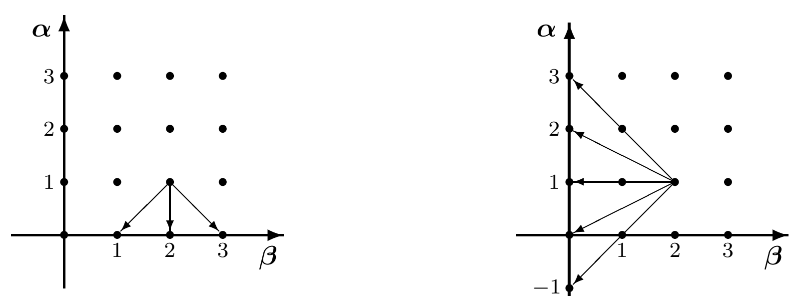
$$(k^2)^{-\alpha} \int d^Dl\ \frac{N}{P^{n_2}_{k+l,m_2} P^{n_3}_{l,m_3}}\ , \tag{42}$$

for the l-integral, where N is a homogeneous polynomial of degree 2α in k^2, $k\cdot l$, l^2, $k\cdot p$ and p^2. After rewriting the whole integral

$$\int d^Dk\ d^Dl\ \frac{(k^2)^{\beta-\alpha}\ N}{P^{n_1}_{k+p,m_1} P^{n_2}_{k+l,m_2} P^{n_3}_{l,m_3}} \tag{43}$$

in terms of T's again we have reduced $T_{\alpha,\beta,n_1,n_2,n_3}$ to integrals $T_{\alpha',\beta',n_1',n_2',n_3'}$ with $\alpha' = 0$, $\beta' \in \{\beta-\alpha,\ \dots,\ \beta+\alpha\}$, $n_1' = n_1$, $n_2' = n_2$, $n_3' = n_3$ or $n_1' + n_2' + n_3' < n_1 + n_2 + n_3$.

This procedure can be illustrated by the following graphical picture. For each triplet (n_1, n_2, n_3) identify the integrals $T_{\alpha,\beta,n_1,n_2,n_3}$ with the points of an associated two-dimensional (α, β)-lattice and order the lattices according to $n_1 + n_2 + n_3$. Then the point (α, β) in the (n_1, n_2, n_3)-lattice can be 'projected' to the points $(0, \beta-\alpha)$, …, $(0, \beta+\alpha)$ on the β-axis and points in lower lying lattices. Analogously, by changing the flow of the external momentum through the diagram, one can project a point on the α-axis.



On the one hand, these projections show that any $T_{\alpha,\beta,n_1,n_2,n_3}$ can be expressed in terms of integrals on the positive axes ($\alpha = 0$ or $\beta = 0$ and $\min(\alpha,\beta) \geq 0$) and trivial one-loop-products, so that at most the positive axes are independent. On the other hand, they entail further relations for fixed (n_1, n_2, n_3):

- First, there are two possibilities to project the points (α, α) on the diagonal onto a positive axis, which sets up a relation between the point $(2\alpha, 0)$ and the points $(\alpha', 0)$ with $0 \le \alpha' < 2\alpha$ and $(0, \beta)$ with $0 \le \beta \le 2\alpha$. In other words, the even points on one of the positive axes, e.g. the positive α-axis, are eliminated from the minimal set of T's.

- Second, equating the two projections of the point $(\alpha, \alpha+1)$ in the line below the diagonal (cf. the above figures) yields a relation between the point $(2\alpha+1, 0)$ and the points $(\alpha', 0)$ with $0 \le \alpha' \le 2\alpha$, $(0, \beta)$ with $0 \le \beta \le 2\alpha+1$ and $(-1, 0)$. The integral $(-1, 0)$, which is of a different topology, can be expressed in terms of $(1, 0)$, $(0, 0)$ and $(0, 1)$ by equating the two projections of $(0, 1)$. In this way the odd points of the positive α-axis with the exception of $(1, 0)$ are eliminated from the minimal set of T's.

The point $(1, 0)$ can be eliminated as well by equating the two projections of $(2, 1)$ and using the above results for $(3, 0)$ and $(2, 0)$. But, unlike the relations for the points $(\alpha, 0)$ with $\alpha > 1$, the relation found in this way contains a denominator $(m_1^2 - p^2)(m_2^2 - m_3^2)$ and is therefore not applicable in all mass cases. For this reason, we prefer to keep $T_{1,0,n_1,n_2,n_3}$ in our set of basic integrals, along with the integrals T_{0,β,n_1,n_2,n_3} with $\beta \ge 0$.

5 Conclusion

In this paper we presented a method to calculate the dimensionally regularized sunset integrals $T_{\alpha,\beta,n_1,n_2,n_3}(m_1^2, m_2^2, m_3^2; p^2)$, see (5), with arbitrary nonzero masses and arbitrary momentum p^2, both below and above the threshold. The simple topology of such diagrams – they can only be cut in one way – allows their divergences to be extracted by subtracting a finite number of terms of their Taylor expansion in p^2 around $p^2 = 0$. The subtracted terms are vacuum diagrams that are known analytically. The remainder, which is finite in $D = 4$ dimensions, is transformed into a well-behaved one-dimensional integral that can be evaluated numerically. Furthermore, we discussed a set of linear relations between different $T_{\alpha,\beta,n_1,n_2,n_3}$'s and suggested a reduction mechanism to a basic set of these integrals, namely $T_{1,0,n_1,n_2,n_3}$ and T_{0,β,n_1,n_2,n_3} with $\beta \ge 0$.

The algorithm was developed in order to meet the needs of a calculation in SU(3) chiral perturbation theory, but it can be applied in any field theory describing massive particles with four-point interaction vertices. The basic idea of the method can be extended to any two-loop diagram containing a subgraph which itself corresponds to a two-point function [10]; in particular, the full three-point graph (1) can be handled in this way for arbitrary momentum transfer q. For chiral perturbation theory the techniques presented in this paper are a step towards taking into account the full mass dependence of the nontrivial two-loop diagrams which occur at the order $\mathcal{O}(p^6)$ of the momentum expansion.

We thank D.J. Broadhurst for some valuable suggestions, J. Gasser for drawing our attention to ref. [21], and K. Schilcher for helpful discussions. J.B.T. was supported by the Graduiertenkolleg “Elementarteilchenphysik bei mittleren und hohen Energien” in Mainz and P.P. by the “Studienstiftung des deutschen Volkes”.

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